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Armington with Heterogeneous Firms

Peter Stephensen

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Peter Stephensen, DREAM (v0.4)

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In applied models, the modelling of price formation and market equilibrium at the sector level, is often based on Armington (1969) and Dixit & Stiglitz (1977). The Dixit-Stiglitz-specification leads to markup pricing:

$$P = (1+m)P^O$$

where P is the market price of a sector, m is an exogenous markup and P^O is the unit cost. If a tax on one of the sectors's inputs is increased (think, for example, of a CO2 tax), the optimal unit costs P^O will increase. For given markup, this will lead to a corresponding relative increase in the output price P. Given the Armington assumption, this price increase will lead to a decline in the sectors exports. If the export elasticity is high (which it typically is in the long run) this will lead to a significant decrease in the sectors's production.

These assumptions are often characterized as being too simplistic. It is argued that the sector in fact consists of many heterogeneous firms. A higher tax must be expected to force the firms with the lowest productivity out of the market, whereas the firms with high productivity have enough profit to adapt to the higher tax.

Melitz (2003) and Hopenhayn (1992) describes a situation with heterogeneous firms. The key subject for Melitz is trade, and especially the fact that it is the highly productive firms that become exporters. Hence, Melitz assumes that the firm charges different prices in the domestic market and in the export markets (pricing-to-market). Our objective is different however, and Armington's assumption of law-of-one-price is maintained. Thereby we are more in line with Hopenhayn (1992). Instead of monopolistic competition (generated by Love-of-Variety) as in Melitz (2003) and Krugman (1985), we assume decreasing returns to scale, so that the concavity of the profit function is explained by technology and not market conditions. As a result, we can assume perfect competition and that the individual firm is a price taker instead of a price setter. The model below can therefore be described as an Armington model with heterogeneous firms.

Below the following alternative price formation is derived:

$$P = \xi \left(P^O\right)^{1-\frac{1}{k}} f^{\frac{1}{k}} \tag{1}$$

where ξ is a constant, f is exogenous entry costs and $k > 1$ is a parameter from the pareto distribution that describes the productivity heterogeneity of firms. The object

$1/k$ describes the degree of heterogeneity. Sectors with more heterogeneity will respond less to an increase in unit costs P^O. This corresponds exactly to the above intuition.

It is also shown that the sector's total pure profit Π is given by:

$$\Pi = \frac{1}{k} PY \tag{2}$$

This is a nice and simple equation that explains the existence of pure profit with heterogeneity.

In addition to the intuitive equations (1) and (2), it can be shown that the assumption of decreasing returns to scale (instead of monopolistic competition as in Krugman (1985) and Melitz (2003)) implies that there are analytical solutions for most of the model variables. This makes the specification very suitable for use in an applied setting. In applied models, as mentioned, markup pricing is often assumed. As an example, consider a CES specification with markup-pricing. This provides the following equation system:

$$X_i = \mu_i \left(\frac{p_i}{P^O}\right)^{-E} Y, i = 1,..,M \tag{3}$$

$$P^O Y = \sum_{i=1}^{M} p_i^x X_i \tag{4}$$

$$P = (1+m) P^O \tag{5}$$

where E is the elasticity of substitution between inputs and μ_i are parameters in the CES function. For given output Y (determined from the demand side) and input prices $(p_1^x,..,p_M^x)$, input quantities X_i are determined by (3). The unit cost P^O is determined in (4). Finally, the output price P is determined in (5).

The next section shows that our new setup leads to this alternative system:

$$X_i = \mu_i \left(\frac{p_i}{P^O}\right)^{-E} X, i = 1,..,M \tag{6}$$

$$P^O X = \sum_{i=1}^{M} p_i^x X_i \tag{7}$$

$$P^O X = \left(1 - \frac{1}{k}\right) PY \tag{8}$$

$$P = \xi \left(P^O\right)^{1-\frac{1}{k}} f^{\frac{1}{k}} \tag{9}$$

For given CES input-aggregate X, inputs X_i is determined by (6). Unit costs P^O are determined in (7). According to (8), the total cost $P^O X$ is given by a fixed share of the revenue PY. This reflects the profit equation (2). Finally, the output price P i is determined by (9).

Observe that from (8) and (9) we have that:

$$\frac{Y}{X} = \frac{k}{k-1}\frac{P^O}{P} = \frac{k}{k-1}\frac{1}{\xi}\left(\frac{P^O}{f}\right)^{\frac{1}{k}}$$

This implies that the "input-productivity" Y/X is endogenous. A higher tax on inputs (leading to higher P^O) will cause this productivity to increase. This fits the intuition that a sector's productivity grows if you tax its inputs. The explanation is that low-productivity firms are leaving the sector.

Another interesting feature of the new model is seen by rewriting (8):

$$PY = \frac{k}{k-1} P^O X$$

According to this equation, the total revenue is given by a fixed markup over the total cost. With this interpretation, we have a markup theory based on heterogeneity. However, this is a value markup and not a price markup. Such a markup can be seen as an example of an *emergent phenomenon*. The value markup cannot easily be traced back to individual behavior, but arises through interaction between the agents of the system. The same can be said about the fixed profit rate in (2).

The significant new parameter is the degree of heterogeneity $1/k$. It is included in the model as a new elasticity explained by heterogeneity. It should be fairly easy to estimate on macro data.

1 The Model

We describe a sector consisting of many heterogeneous firms. The firms are price-takers. The individual firm has an S-shaped production function, as it has increasing returns to scale at low production and decreasing returns to scale at high production. The firm therefore has an optimal size for given market price.

The firms are heterogeneous in terms of total productivity. It is shown below that high-productivity firms have higher profit than low-productivity firms. Firms that have a negative profit at the current market price (because of low productivity) exit the market. New entrants pay an exogenous entry cost to enter the market. It then draws its productivity from a pareto distribution. The market price ensures that new firms have an expected profit corresponding to the entry cost.

The firm's production function is given by

$$y = \varphi \left(max \left\{ x^{\alpha} - \phi, 0 \right\} \right) \tag{10}$$

where $0 < \alpha < 1$, $\varphi > 0, \phi > 0$ and where

$$x = F\left(x_1, .., x_M\right)$$

The function F is assumed to exhibit constant returns to scale. One can think of F as a homogeneous input aggregator. The parameters φ, ϕ and α describe productivity, the degree of increasing returns to scale and the degree of decreasing returns to scale, respectively. Note that the production function (10) is the same as in Melitz (2003) for $\alpha = 1$.

For a given value of x, the firm will minimize its costs $C = \sum_j p_j^x x_j$. Since the F-function has constant returns to scale, we know that there is a price index P^O so that the optimal costs are given by:

$$C = P^O x$$

It therefore applies from (10) that

$$C = P^O \left(\frac{y}{\varphi} + \phi \right)^{\frac{1}{\alpha}}$$

The firms are heterogeneous in terms of productivity φ. New entrants are assumed to be able to enter the market if they pay an exogenous annuity entry cost f. The firm then draws its productivity from a pareto distribution with the density function $g(\varphi)$:

$$g(\varphi) = k \frac{\varphi_0^k}{\varphi^{k+1}}, k > 1, \tag{11}$$

where φ_0 is the minimal productivity. The corresponding accumulated distribution is given by

$$G(\varphi) = 1 - \left(\frac{\varphi_0}{\varphi} \right)^k \tag{12}$$

Market price

The sector's market price P should ensure that the expected net gain upon entering the market is 0. That means that

$$E[\pi] = prob[\text{entry}] \cdot E[\pi|\text{entry}] = f \tag{13}$$

where $prob[\text{entry}]$ is the probability that the new firm has sufficiently high productivity to enter the market and $E[\pi|\text{entry}]$ is the expected profit if a firm enters the market. The entry cost is given by f.

Given the market price P, the firm's profit is given by:

$$\pi = Py - P^O \left(\frac{y}{\varphi} + \phi \right)^{\frac{1}{\alpha}} \tag{14}$$

The first order condition with regard to production y is given by:

$$P = \frac{1}{\alpha} \frac{P^O}{\varphi} \left(\frac{y}{\varphi} + \phi \right)^{\frac{1}{\alpha} - 1} \tag{15}$$

The optimal firm size is therefore given by:

$$y = \varphi^{\frac{1}{1-\alpha}} \left(\alpha \frac{P}{P^O} \right)^{\frac{\alpha}{1-\alpha}} - \varphi\phi \tag{16}$$

From (14) and (16) we have that

$$\pi = P \left((1-\alpha) \left(\varphi\alpha \frac{P}{P^O} \right)^{\frac{\alpha}{1-\alpha}} - \phi \right) \varphi \tag{17}$$

In (17) there is a unique productivity $\hat{\varphi}$ that ensures zero profit:

$$\hat{\varphi} = \left(\frac{\phi}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha}} \frac{1}{\alpha}\frac{P^O}{P} \tag{18}$$

It is obvious from (17) that profit is increasing in productivity. A potential new firm will therefore enter the market if $\varphi \geq \hat{\varphi}$.

We can now calculate $prob\,[\text{entry}]$ og $E\,[\pi|\text{entry}]$ such that equilibrium condition (13) can be defined. From (12) we have that

$$prob\,[\text{entry}] = prob\,[\varphi \geq \hat{\varphi}] = \left(\frac{\varphi_0}{\hat{\varphi}}\right)^k$$

In addition we have that

$$E\,[\pi|\text{entry}] = E\,[\pi|\varphi \geq \hat{\varphi}] = \int_{\hat{\varphi}}^{\infty} \pi\,(\varphi)\,\mu\,(\varphi)\,d\varphi$$

where

$$\mu\,(\varphi) \equiv \frac{g\,(\varphi)}{1-G(\hat{\varphi})} = k\frac{\hat{\varphi}^k}{\varphi^{k+1}}$$

Simple integral calculus implies that for all $a < k$:

$$\int_{\hat{\varphi}}^{\infty} \varphi^a \mu\,(\varphi)\,d\varphi = \frac{k}{k-a}\hat{\varphi}^a \tag{19}$$

From (17) and (19) it can then be calculated that

$$E\,[\pi|\varphi \geq \hat{\varphi}] = P\left((1-\alpha)\left(\alpha\frac{P}{P^O}\right)^{\frac{\alpha}{1-\alpha}} \frac{k}{k-\frac{1}{1-\alpha}}\hat{\varphi}^{\frac{1}{1-\alpha}} - \phi\frac{k}{k-1}\hat{\varphi}\right)$$

Substitute (18) to get:

$$E\,[\pi|\varphi \geq \hat{\varphi}] = \frac{k}{k-1}\frac{1-\alpha}{(1-\alpha)k-1}\left(\frac{\phi}{1-\alpha}\right)^{\frac{1}{\alpha}} P^O \tag{20}$$

From (13) we then have that

$$P = \xi\left(P^O\right)^{1-\frac{1}{k}} f^{\frac{1}{k}} \tag{21}$$

where

$$\xi \equiv \left(\frac{k}{k-1}\frac{1-\alpha}{(1-\alpha)k-1}\left(\frac{\phi}{1-\alpha}\right)^{\frac{1}{\alpha}}\right)^{-\frac{1}{k}} \frac{1}{\alpha}\left(\frac{\phi}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha}} \frac{1}{\varphi_0} \tag{22}$$

Hereby we have shown the price equation (21).

Number of firms

The number of firms can be calculated if we know the average firm size $\overline{y}$:

$$n = \frac{Y}{\overline{y}} \tag{23}$$

So let's calculate the average firm size:

$$\overline{y} = \int_{\hat{\varphi}}^{\infty} y(\varphi)\,\mu(\varphi)\,d\varphi$$

By using (16) and the rule (19) we get:

$$\overline{y} = \left(\alpha \frac{P}{P^O}\right)^{\frac{\alpha}{1-\alpha}} \frac{k}{k - \frac{1}{1-\alpha}} \hat{\varphi}^{\frac{1}{1-\alpha}} - \phi \frac{k}{k-1} \hat{\varphi}$$

Substituting $\hat{\varphi}$ from (18) yields:

$$\overline{y} = \frac{1}{\alpha} \frac{P^O}{P} \left(\frac{k}{(1-\alpha)k - 1} - \frac{k}{k-1} \right) (1-\alpha) \left(\frac{\phi}{1-\alpha} \right)^{\frac{1}{\alpha}} \tag{24}$$

Substituting the price equation (21) gives:

$$\overline{y} = \lambda \left(\frac{P^O}{f} \right)^{\frac{1}{k}}$$

where

$$\lambda \equiv \phi \varphi_0 \left(\frac{k}{(1-\alpha)k-1} - \frac{k}{k-1} \right) \left(\frac{k}{k-1} \frac{1-\alpha}{(1-\alpha)k-1} \left(\frac{\phi}{1-\alpha} \right)^{\frac{1}{\alpha}} \right)^{\frac{1}{k}}$$

From (23) we then have that:

$$n = \frac{1}{\lambda} \left(\frac{f}{P^O} \right)^{\frac{1}{k}} Y \tag{25}$$

For given unit costs P^O, the sector's adjustment to total sector demand Y will take place on the extensive margin, as the number of firms will adapt. Adjustment to the intensive margin occurs only if unit costs change. In this case, there is both a change in the individual firm's production and a change in the sector's productivity distribution. The effect will therefore be both individual and systemic.

Total sector profits and cost

Total profits in the sector is given by:

$$\Pi = n \cdot E\left[\pi \mid \varphi \geq \hat{\varphi}\right] \tag{26}$$

From (20) and (26) we have that:

$$\Pi = \frac{k}{k-1} \frac{1-\alpha}{(1-\alpha)k-1} \left(\frac{\phi}{1-\alpha} \right)^{\frac{1}{\alpha}} \frac{1}{\lambda} \left(P^O\right)^{1-\frac{1}{k}} f^{\frac{1}{k}} Y$$

Substitution of the price equation (21) and after tedious calculations we get:

$$\Pi = \frac{1}{k} PY \tag{27}$$

We have by definition that:

$$PY = \Pi + P^O X$$

where X is the total aggregate input of the sector and $P^O X$ is the total cost. We therefore have that

$$P^O X = \left(1 - \frac{1}{k}\right) PY \tag{28}$$

Alternatively, this can be written as:

$$PY = \frac{k}{k-1} P^O X$$

According to this equation, the total revenue is given by a fixed markup over the total cost. With this interpretation, we have a markup theory based on heterogeneity.

2 References

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